Written Comprehensive Examinations

Northeastern University
Department of Mechanical & Industrial Engineering

[ MIE DOCTORAL QUALIFYING EXAMINATIONS GUIDELINES ]

Important information, deadlines, exams selection procedures

Written Comprehensive Examinations
DOCTORAL QUALIFYING EXAMINATIONS

Background and Motivation: To demonstrate breadth and depth in each of the subject exams, cross-over and merging exams are necessary in an effort to provide students with an opportunity to master the core disciplines in mechanical or industrial engineering (at both undergraduate and graduate levels) along with a focus area of importance to their specialization. These exams also provide an assessment as to whether students have adequate knowledge to pursue advanced study, and possess attributes of a doctoral candidate by demonstrating understanding of and the ability to apply fundamental principles. Also, an oral exam tied to the written exams is necessary in an effort to evaluate student’s potential to perform independent research in the chosen field of specialization for the doctoral program.

Doctoral Qualifying Examinations Framework: The Doctoral Qualifying Examinations consist of the following two parts:

1. Two Written Comprehensive exams, which are respectively referred to as major Exam A and minor Exam B.
2. An Oral Area exam equivalently referred to as the Area Exam. This exam can be administered at any time after passing the written comprehensive exams, but no later than the end of the semester in which the written exams are taken and passed.

WRITTEN COMPREHENSIVE EXAMINATIONS

All doctoral students who hold a master’s degree must take the written comprehensive exams no later than the first time that it is offered after their first academic year of study. Those admitted directly with a bachelor’s degree must take the written comprehensive exams no later than the first time that it is offered after their first two years of study. The written comprehensive exams include two (2) exams, Exam A and Exam B; and are given on Thursday and/or Friday of the first week of classes during regular semesters.

Written Comprehensive Exams Rules: Exam A, about 4-6 hours in length, should be selected from the list of major exams based on the student’s concentration (i.e., Industrial Engineering (IND), Materials (MSE), Mechanics (MEC), Mechatronics (DSC), or Thermofluids (TFS)), see Table 1. No deviation from this rule will be permitted. As listed in Table 1, Exam B, about 1-2 hours in length, is a degree program-dependent exam and should be selected from the list provided for each PhD program in MIE Department (i.e., PhD degree program in Industrial Engineering—IE or PhD degree program in Mechanical Engineering—ME). Only one exam from this list should be selected. All students are required to have their Research Advisor’s approval on selection of Exams A and B prior to registering to take the written comprehensive exams. Note that Exam B cannot be similar or close to one of the topics covered in Exam A.
ORAL AREA EXAMINATION

The objective of the **Oral Exam**, also referred to as **Area Exam**, is to assess the student’s potential to perform independent research in the chosen field of specialization. This exam can be administered at any time after passing the written comprehensive exams, but no later than the end of the semester in which the written exams are taken and passed.

**Oral Area Examination Procedure:** The student’s Research Advisor convenes and chairs an Oral Examination Committee comprised of a minimum of three (3) faculty or affiliated faculty members of the MIE Department deemed appropriate to the student’s research field. This committee provides a set of technical papers pertinent to the student’s research area. The Oral Examination Committee will then conduct the exam, which is comprised of the following two parts (typically a one-hour session):

1. an oral presentation of at least 30-min on a select number of papers out of the assigned technical papers, and
2. an oral exam of ~30-min by committee members’ questions and evaluation of the student covering topics specifically related to the student’s research area.

GRADING PROCEDURE

**Grading Procedure and Results of the Written Comprehensive Examination:** The MIE Graduate Affairs Committee (GAC) will review all students’ performance in the written comprehensive exams. Depending on the results of both major and minor exams and in consultation with the student’s Research Advisor, the GAC will recommend one of following three possible options:

1. **No invitation to oral area exam:** The student will be dismissed from the program. He/she may be granted an MS degree if the requirements are already met; otherwise, the student may continue to fulfill the requirements for an MS degree in industrial engineering (IE), mechanical engineering (ME), or operations research (OR).
2. **No invitation to oral area exam yet:** The student will be asked to re-take the written exam(s) again in the next offering; and/or take additional courses.
3. **Student is invited to oral area exam.**

**Grading Procedure and Results of the Oral Area Examination:** If the performance of the student in oral area exam is not satisfactory, the student will be dismissed from the program. He/she may be granted an MS degree if the requirements are met; otherwise, the student may continue to fulfill the requirements for an MS degree in industrial engineering (IE), mechanical engineering (ME), or operations research (OR).

Upon successfully passing the oral exam, the student continues in the PhD program and in case of passing all the required coursework, he/she will become a PhD Candidate. The results of the written and oral exams and any recommended coursework become part of the student’s record.
## DETAILED LIST OF EXAMS A & B

**Table 1:** List of Exams A* and B based on student’s concentration.

<table>
<thead>
<tr>
<th>Concentration</th>
<th>Exam A</th>
<th>Exam B</th>
</tr>
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</table>
| Industrial    | Industrial Engineering (IND) | **Exams B for IE PhD Students:**  
1. Data Mining (DMN)  
2. Human-Machine Systems (HMS)  
3. Manufacturing Systems (MFS)  
4. Networks and Advanced Optimization (NAO)  
5. Reliability and Quality Assurance (RQA)  
6. Supply Chain Engineering (SCE) |
| Materials     | Materials Science Engineering (MSE) |  |
| Mechanics     | Mechanics (MEC) | **Exams B for ME PhD Students:**  
1. Control Systems (DSC3)  
2. Dynamic Systems (DSC1)  
3. Dynamics and Vibration (MEC2)  
4. Electric/Magnetic/Optics (EMO)  
5. Engineering Mathematics (MTH)*  
6. Finite Element Method (MEC3)  
7. Fluid Mechanics (TFS2)  
8. Heat Transfer (TFS3)  
9. Mechanics of Deformable Media (MEC1)  
10. Soft Materials (SM)  
11. Thermodynamics (TFS1) |
| Mechatronics  | Dynamic Systems and Control (DSC) |  |
| Thermofluids  | Thermofluids Science (TFS) |  |

* List of Exams A

- **Industrial Engineering (IND):** Probability (IND1), Statistics (IND2), and Deterministic OR (IND3).
- **Materials Science Engineering (MSE):** Kinetics of Materials (MSE1), Thermodynamics of Materials (MSE2), and Diffusion, Soft Matter, and Mechanical Behavior (MSE3)
- **Mechanics (MEC):** Mechanics of Deformable Media (MEC1), Dynamics and Vibration (MEC2), and Finite Element Method (MEC3)
- **Dynamic Systems and Control (DSC):** Dynamic Systems (DSC1), Mechanical Vibrations (DSC2), and Control Systems (DSC3)
- **Thermofluids Science (TFS):** Thermodynamics (TFS1), Fluid Mechanics (TFS2), and Heat Transfer (TFS3), and Engineering Mathematics (TFS4) – *TFS area students should choose three out of the four from this list for Exam A.*

†Engineering Mathematics (MTH) is **REQUIRED** as EXAM B for all students taking Exam A in Mechanics (MEC) area.
EXAMS A & B: TOPICAL COVERAGE/DETAILS

Important information for exam content
Engineering Mathematics (MTH) – Exam B

MTH:

- Ordinary differential equations using exact methods, series and transforms.
- Partial differential equations using separation of variables (Fourier series, eigenfunction expansions) and transforms.
- Linear algebra matrices and linear equations, determinants, eigenvalue problems.
- Vector field theory including Cartesian, cylindrical and spherical coordinates, gradient, divergence and curl, and integral theorems (Divergence Theorem, Stokes’ Theorem).

Suggested References for Preparation:


**Previous Exam problems for MTH provided in the Appendix at the end of this document.**
Industrial Engineering (IND) – Exam A

IND1: Probability:
- Discrete and continuous random variables.
- Cumulative probability distributions and moment generating functions.
- Expectation of random variables.
- Discrete and continuous probability distributions including: binomial, Poisson, geometric, uniform, exponential and normal.
- Multivariable probability distributions, covariance and independence of random variables.
- Sampling distributions and limiting theorems.
- Parameter estimation.
- Confidence intervals and hypothesis testing.
- Regression and ANOVA.
- Chi-squared and non-parametric tests.

IND2: Statistics:
- Regression and ANOVA.
- Chi-squared and non-parametric tests.
- Stochastic Processes
- Poisson Process and Exponential Distribution
- Markov Chains (Discrete/Continuous time Markov Chain)
- Birth and Death process
- Queuing Theory

IND3: Deterministic OR:
- Linear Programming (LP)
  - Formulation of LP models
  - Solution of LP models with the graphical method and the simplex algorithm
  - Theory of the simplex method
  - Duality theory and dual simplex algorithm
  - Sensitivity Analysis in LP
• The Transportation Problem and the Hungarian algorithm

• Network Optimization Models
  o The Shortest Path Problem
  o The Minimum Spanning Tree Problem
  o The Maximum Flow Problem
  o The Minimum Cost Flow Problem

• Dynamic Programming (DP)
  o Discrete State DP problems
  o Continuous State DP problems

Suggested References for Preparation:


Exam B Listings for Industrial Engineering (IND) Area

**Data Mining (DMN):**

- Core concepts of data mining and predictive modeling
- Data visualization for exploration and decision making
- Dimension reduction and data curation
- Feature extraction and feature selection
- Evaluating predictive performance of machine learning models
- Multiple linear regression
- k-nearest neighbors for classification and regression
- Naïve Bayes classifier
- Classification and regression trees
- Logistic regression
- Neural networks (multilayer feed-forward neural networks)
- Support vector machine models
- Linear discriminant analysis
- Association rules and collaborative filtering
- Cluster analysis
- Time series analysis

**Suggested References for Preparation:**


**Human-Machine Systems (HMS):**

- Sociotechnical Systems and Human Systems Engineering, Human Capabilities and Characteristics.
- Engineering Anthropometry and Biomechanics.
• Physiology related to Human Factors and Workstation Design.
• Taxonomy of Biosensors for various cues (psychological, physiological, physical), states and behaviors of humans.
• Basic principles of biosensors, current technologies for building biosensors.
• Cognition and Information Processing, Decision-Making, Attention and Workload.
• Human-Machine Interface Design, Controls and Displays.
• Safety Engineering, Human Error and Accident analysis.
• Human Factors in Transportation, Automation.
• Human-Robot Interaction and human friendly mechatronics.
• Human Factors in Healthcare and Patient Safety, Human Factors and Ergonomics in Manufacturing and Service Industries.

Suggested References for Preparation:


**Manufacturing Systems (MFS):**

• Manufacturing operations
• Manufacturing metrics and economics
• Elements of manufacturing systems
• Single-station manufacturing cells
• Manual assembly lines
• Automated production lines
• Automated assembly systems
• Group Technology and cellular manufacturing
• Flexible Manufacturing cells and system
• Mechanical properties of materials
• Fundamentals of metal forming
• Bulk deformation processes in metalworking
• Theory of metal machining
• Cutting tool technology

Suggested References for Preparation:


Network and Advanced Optimization (NAO):
• Geometry of Linear Programming
• Revised Simplex
• Duality
• Complementary Slackness
• Representation of Polyhedra
• Solving large scale optimization problems: Column Generation and Constraint Generation (Cutting plane methods)
• Network flows including: trees, assignment and transportation problems, max flow min cut, shortest paths, min cost flow, multi-commodity flow

Suggested References for Preparation:
Reliability and Quality Assurance (RQA):

- Quality planning, Control and Improvement.
- Process control, Discrete and Continuous Control Charts.
- Moving Average and Custom Control Charts.
- Discrete and Variable Sampling Methods, Mil Standards.
- Process Capability Analysis.
- Quality Engineering Method of Robust Design.
- Mathematical Definitions of Reliability, Hazard Rate, Intensity Function, Failure Rate and Availability.
- Stress and Strength Analysis, Reliability Block Design, Fault Tree Method.
- Network Reliability Methods, Markovian Methods, Reliability Testing.
  - Reliability Estimates from Field and Test Data.
  - Confidence Interval on Reliability.
- Maintenance and Replacement Policies.

Suggested References for Preparation:


Supply Chain Engineering (SCE):

- Forecasting
- Aggregate planning
- Sequencing and Scheduling
- Inventory analysis and control
- Materials requirement planning
- Pricing and revenue management
• Manufacturing resource planning
• Project management
• Contracts decisions
• Transportation decisions
• Location and distribution decisions
• Supplier selection methods
• Global supply chains

Suggested References for Preparation:


Materials Science Engineering (MSE) – Exam A

MSE1: Kinetics of Materials:
- Diffusion and Brownian Motion
- Solidification
- Diffusional and diffusionless transformations in solids

MSE2: Thermodynamics of Materials:
- Three laws
- State functions
- Systems
- Phase equilibria and stability
- Behavior of Solutions
- Relations to phase diagrams
- Reactions among condensed phases and gases
- Statistical Thermo (entropy, heat capacity, etc.)

MSE3: Diffusion/Soft Matter/Mechanical Behavior:
- Molecular/colloidal forces
- Crystal structures
- Dislocation theory
- Mechanical Behavior and strengthening mechanisms
- Fatigue and fracture

Suggested References for Preparation:


Exam B Listings for Materials Science Engineering (MSE) Area

**Soft Materials (SM):**
- Kinetics of Molecules and Colloids (including drift, diffusion, etc.)
- Thermodynamics of Soft Interfaces (includes surface tension, adsorption, Laplace pressure, etc.)
- Polymers (processing, structure, properties, etc.)

**Suggested References for Preparation:**


**Electric/Magnetic/Optics (EMO):**
- Electronic and Magnetic Materials
- Devices
- Functional Materials

**Suggested References for Preparation:**

Mechanics (MEC)

Engineering Mathematics (MTH) is REQUIRED for Exam B for ALL students taking the Exam A in Mechanics (MEC) area.

MEC1: Mechanics of Deformable Bodies:
- Basic concepts of stress and strain and stress-strain relations.
- Yield strength and elastic-perfectly-plastic material behavior.
- Transformation of stress, principal stresses in three dimensions, Mohr’s circles.
- Boundary and continuity conditions for three-dimensional continua.
- Structural mechanics of bars, shafts, and beams under axial, torsional, and transverse loadings.
- Energy methods (including Castigliano’s Second Theorem) and calculation of deflections (including shear deformation) of beams, frames, and rings for statically determinate and statically indeterminate loadings.
- Thin-walled pressure vessels.
- Stability of structures; buckling of columns and structures.

MEC2: Dynamics and Vibration:
- Basic concepts of rigid body kinematics and kinetics.
- Newton’s Laws of Motion.
- Energy and momentum methods for particles and rigid bodies.
- Free and forced vibrations of single and multiple degree-of-freedom systems.
- Eigenvalue problems and modal expansions in vibration.
- Simple vibration of rods (longitudinal and torsional) and beams (bending).

MEC3: Finite Element Method:
- Weighted residual methods and identification of essential and natural boundary conditions.
- Implementation of variational methods, such as Rayleigh-Ritz, Galerkin to 1D boundary value problems.
- Derivation of interpolation functions and Cn continuity.
- Truss, beam, and 2D solid elements and element defects.
• Global stiffness matrix, assembly of element equations, numerical implementation of boundary conditions.
• Isoparametric elements and numerical integration.
• Solution of transient problems with implicit and explicit methods.
• Application of finite element method in heat transfer problems.

Suggested References for Preparation:

MEC1:

MEC2:

MEC3:
Dynamic Systems and Control (DSC)

DSC1: Dynamic Systems

- Dynamics of Mechanical Systems (Translational and Rotational)
- Electrical Circuits and Op-Amps
- Electromechanical Devices and DC Motors
- Thermal Systems Modeling
- Fluid and Level Systems Modeling
- Linearization Techniques
- Laplace Transforms and Application to Dynamic Systems Analysis
- 1st-order and 2nd-order Systems Response Characteristics: Time Constants; Natural Frequency, Damping Ratio, Damped Frequency; Impulse, Step and Ramp Responses; and Steady-state Error
- Approximating Higher-order Systems with equivalent 1st- and 2nd-order System with Dominant Modes

DSC2: Mechanical Vibrations

- Free and Forced Vibrations of Single Degree-Of-Freedom (SDOF) and Multiple Degree-Of-Freedom (MDOF) Systems
- Damped Vibrations
- General Eigenvalue Problem and Modal Analysis/Expansion in Vibrations
- Equations of Motion and Boundary Conditions for Transverse Vibrations of Strings; Longitudinal Vibrations of Bars; and Torsional Vibrations of Shafts
- Free and Forced Transverse (Bending) Vibrations of Euler-Bernoulli (Thin) Beams: Mode Shapes and Orthogonality Conditions

DSC3: Control Systems

- System Modeling Diagrams; Block Diagram Algebra and Reduction
- Effects of Poles/Zeros on System Response
- Routh’ Stability Criterion
- The 3-term PID (Proportional, Derivative and Integral) Control Design and Analysis
- Root Locus Plot
- System Type and Analysis of Steady-state Error
Design via Root Locus and Feedback Compensation Techniques
Bode Plots
Nyquist Plot and Nyquist Stability Criterion
Gain and Phase Margins
Design via Frequency Response (Lead and Lag Compensations).
Standard Forms (Input/Output) and State Equations
State-Space, Observability, Controllability, Control Canonical Form, Pole Placement, Ackerman Formula
Introduction to Real-time Control Implementation

Suggested References for Preparation:

**DSC1:**

**DSC2:**

**DSC3:**

## Thermofluids Science (TFS)

TFS students should choose three (3) out of four (4) areas listed below for Exam A. *Note: MTH cannot be taken as Exam B if TFS4 is chosen as a part of Exam A.*

### TFS1: Thermodynamics
- Conservation of mass (steady and transient)
- Steady and Transient First and Second law of thermodynamics
- Energy, available energy and entropy
- Temperature and pressure
- Work and heat interaction
- Heat engine
- Characteristic function
- Simple system
- Equation of state
- Conversion devices
- Power generation
- Refrigeration and energy pump
- Chemical reaction and chemical equilibrium

### TFS2: Fluid Mechanics
- Fluid Statics
- Finite control volume analyses
  - Reynolds transport theorem; continuity
  - Momentum Principle; conservation of energy
  - Bernoulli equation
- Differential control volume analyses
  - Navier-Stokes equations; laminar flow analyses
  - Boundary layer analyses; potential flow analyses
- Similitude
  - PI theorem; non-dimensional parameters
- Model flow vs. prototype flow
- Non-dimensionalization of governing equations and boundary conditions

**TFS3: Heat Transfer**

- **Conduction**
  - Steady and transient heat transfer in multi-dimensional systems (development and solution)
  - Problems involving internal heat generation source
  - Extended surface problems, including fin efficiency
  - Extended surface problems with varying cross section area (Bessel function solutions)
  - Transient problems, lumped capacitance and multi-dimensional systems

- **Convection**
  - Definition of the heat transfer coefficient
  - Use of correlations
  - Hydrodynamic and thermal boundary layer
  - Use of the integral analysis to calculate the heat transfer coefficient
  - Natural convection
  - Application of the Navier-Stokes equations to the convection problem

- **Radiation**
  - Use of the Stefan-Boltzmann Law, Planck's distribution Law and black body emissivity functions
  - Emissivity, reflectivity and transmittance definitions and their use in spectral and gray surfaces
  - Kirchoff's Law and multi-band width problems
  - Definition and use of view factors
  - Solution of multi-surface problems
  - Combined modes of heat transfer problems

**TFS4: Engineering Mathematics (MTH):**

- Ordinary differential equations using exact methods, series and transforms.
- Partial differential equations using separation of variables (Fourier series, eigenfunction expansions) and transforms.
• Linear algebra matrices and linear equations, determinants, eigenvalue problems.
• Vector field theory including Cartesian, cylindrical and spherical coordinates, gradient, divergence and curl, and integral theorems (Divergence Theorem and Stokes’ Theorem).

Suggested References for Preparation:

Problem 1 (10 Pts)
Suppose a matrix, \( M \), is 3-by-3, real, and symmetric and has eigenvectors \( \psi_1, \psi_2, \) and \( \psi_3 \) with corresponding eigenvalues \( \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3 \). Suppose the vector \( x = 2\psi_1 - 2\psi_2 \). If \( M^4 x = a\psi_1 + b\psi_2 + c\psi_3 \), what are \( a, b, \) and \( c \)?

Problem 2 (20 Pts)
A circular disk of radius \( a \) has a mass per unit area:

\[
\rho = \rho_0 \frac{y^2}{x^2 + y^2} e^{-\left(\frac{x^2 + y^2}{\lambda^2}\right)}
\]

where the center of the circle is at \( x = 0, y = 0 \), \( \lambda \) is a constant with units of length, and \( \rho_0 \) is a constant with units of mass per unit area. Calculate the total mass and leave your result in terms of \( a, \rho_0, \) and \( \lambda \).

Problem 3 (20 Pts)
A scalar field in 2 dimensions has the form:

\[
\phi(x, y) = \log[x^2 + y^2]
\]

What is the integral of the Laplacian, \( \nabla^2 \phi = \nabla \cdot (\nabla \phi) \), over the region defined by \( x^2 + y^2 \leq 1 \)? (Hint: you should use Gauss’s divergence theorem.)

Problem 4. (50 Pts): The temperature in a disk of radius \( a = 5\text{cm} \) is governed by the diffusion equation

\[
\frac{\partial T}{\partial t} = \alpha^2 \nabla^2 T
\]

It is initially at a spatially uniform temperature of 100 K, and its edge is then quenched to a thermal bath fixed at \( T = 0 \text{K} \). Solve for temperature distribution in the disk, \( T(x, y, t) \). The thermal diffusivity of the disk is \( \alpha^2 = 10^{-4}\text{m}^2/\text{s} \). Assume the top and bottom faces of the disk are insulated. Note: It is acceptable to express the answer in terms known functions or transforms.
Appendix

The Diffusion Equation

The governing equation for thermal diffusion is of the form

$$\alpha^2 \nabla^2 T = \frac{\partial T}{\partial t},$$  \hspace{1cm} (1)

where $T$ is the temperature, with

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \hspace{1cm} (2)$$

$$\equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \hspace{1cm} (3)$$

$$\equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left( \frac{\partial^2}{\partial \phi} \right) \hspace{1cm} (4)$$

in the cartesian, cylindrical and spherical coordinates, respectively.

Some useful second order ordinary differential equations

Euler-Cauchy Equations

An Euler-Cauchy equation is of the form

$$x^2 y'' + b xy' + cy = 0 \hspace{1cm} (5)$$

where $b$ and $c$ are constant numbers. These equations can be solved by using the change of variable $x = e^t$.

Legendre Equation

The Legendre differential equation is the second-order ordinary differential equation

$$(1 - x^2)y'' - 2xy' + l(l + 1)y = 0. \hspace{1cm} (6)$$

The equation has two linearly independent solutions, a solution $P_l(x)$ which is regular at finite points is called a Legendre function of the first kind, while a solution $Q_l(x)$ which is singular at $\pm 1$ called a Legendre function of the second kind. If $l$ is an integer, the function of the first kind reduces to a polynomial known as the Legendre polynomial. The first few Legendre polynomials
are

\[
P_0(x) = 1 \\
P_1(x) = x \\
P_2(x) = \frac{1}{2}(3x^2 - 1) \\
P_3(x) = \frac{1}{2}(5x^3 - 3x) \\
P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \\
P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \\
P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5).
\]

If the variable \(x\) is replaced by \(\cos \theta\), then the Legendre differential equation becomes

\[
\frac{d^2 y}{d\theta^2} + \frac{\cos \theta}{\sin \theta} \frac{dy}{d\theta} + l(l+1)y = 0.
\] (7)

**Bessel Equation**

The differential equation

\[
x^2 y'' + xy' + (x^2 - n^2)y = 0
\] (8)

has two classes of solution, called the Bessel function of the first kind \(J_n(x)\) and Bessel function of the second kind \(Y_n(x)\). It follows then \(J_n(kx)\) and \(Y_n(kx)\) are solutions to the equation

\[
x^2 y'' + xy' + (k^2 x^2 - n^2)y = 0
\] (9)

Some useful Bessel identities are as follows:

\[
\frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x),
\]

\[
\int_0^c [J_n(kx)]^2 x dx = \frac{c^2}{2} [J_{n+1}(kc)]^2
\]

**Laplace Transform**

The Laplace transform of a function \(f(t)\), defined for all real numbers \(t \geq 0\), is the function \(F(s)\), defined by

\[
F(s) = \int_0^\infty f(t)e^{-st} \, dt
\]
Problem 1 (20 Pts)
Consider Laguerre’s equation:
\[ xL''_n + (1 - x)L'_n + nL_n = 0 \]

a) Derive the recursion relation for a series solution about \( x=0 \).
b) Show that when \( n \) is a non-negative integer, the recursion has a polynomial solution which terminates after a finite number of terms, and explicitly construct the first 3 of these polynomials, \( L_0, L_1, \) and \( L_2 \).

Problem 2 (40 Pts)
Consider the solid paraboloid of revolution described by:
\[ x^2 + y^2 \leq z \]
\[ 0 \leq z \leq 1 \]
Consider the vector field:
\[ \vec{v} = (x^3 + xy^2)\hat{i} + (y^3 + x^2y)\hat{j} \]
where, as usual, \( \hat{i} \) and \( \hat{j} \) are the unit vectors pointing along the \( x \) and \( y \) Cartesian axes.

a) Compute the integral over the boundary of \( \vec{v} \cdot \hat{n} \) where \( \hat{n} \) is the outward unit normal vector.
b) Compute the volume integral of the divergence of \( \vec{v} \) to explicitly verify the divergence theorem.

Problem 3 (40 Pts)
Consider a field \( u(r, \theta) \) governed by the Laplace equation, \( \nabla^2 u = 0 \). Recall that the form of the Laplace operator in polar co-ordinates is:
\[ \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \]

Solve for \( u(r, \theta) \) in a circular region of radius \( a \) subject to the following boundary condition:
\[ u(a, \theta) = T_1 \quad 0 \leq \theta < \pi \]
\[ u(a, \theta) = T_2 \quad \pi \leq \theta < 2\pi. \]
Problem 1: Quadrupole source for Poisson equation (40pts)
Consider an infinite conductor in two dimensions (a sheet). The steady-state temperature profile is governed by the Poisson equation:
\[ \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \sigma \]
where \( T \) is the temperature and \( \sigma \) is the local rate of heating/cooling by heat sources/sinks.
The solution, \( G \), to the Poisson equation with a unit point source located at \( r', \phi' \) has the form:
\[ G(r,\phi; r',\phi') = \frac{\ln(r^2)}{4\pi} - \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{r}{r'} \right)^m \cos[m(\phi - \phi')] \]
where \( r_\text{<} \) is defined to be the lesser of \( r \) and \( r' \) and \( r_\text{>} \) is defined to be the greater of \( r \) and \( r' \).
a) Using the result for a single point source, find the temperature profile in an infinite conducting sheet with four sources; two sources with strength \( \sigma_0 \) located at \( x = \pm a, y = 0 \) and two sinks with strength \(-\sigma_0\) located at \( x = 0, y = \pm a \). You should leave your result expressed as an infinite series, and you should have explicit expressions for all terms in the series.
b) Keeping only the first two non-zero terms in your series solution, find the expression for the Cartesian components of the gradient, \( \nabla T = \hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} \), of \( T \) near the origin (\( r << a \)) in terms of \( x \) and \( y \). Comment on the ratio of the first to second term and how it depends on \( r \).

Problem 2: Diffusion along a rod (40pts)
Thermal diffusion within a thin rod of length \( L \) is found to be of the form
\[ \alpha^2 u_{xx} = u_t - q_0, \quad (0 < x < L, 0 < t < \infty). \]
The rod is initially held at temperature \( T_0 \) and thereafter, the left edge of the rod is held at \( T_1 \) while the right edge is thermally insulated so that \( u_x = 0 \) at \( x = L \). \( q_0 \) is an arbitrary constant.
Find an expression for \( u(x,t) \) in terms of \( \alpha, T_0, T_1, q_0 \) and \( L \). If your final answer is in series form, please find a closed form expression for all coefficients.

Problem 3: Vibrations on a membrane (20pts)
The height, \( u(x,y,t) \), of a rectangular membrane, (with \( 0 < x < a, 0 < y < b \)), is governed by the wave equation,
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \]
with wavespeed, \( c \). The membrane is held at \( u = 0 \) on the sides \( (x = 0, x = a, y = 0, y = b) \) The initial displacement is
\[ u(x,y; t = 0) = u_0 \sin(3\pi x/a) \sin(\pi y/b) \]
where \( u_0 \) is an arbitrary constant and the membrane is released from rest. Solve for \( u(x,y,t) \) in terms of \( u_0, c, a, \) and \( b \). If your final answer is in series form, please find a closed form expression for all coefficients.
Useful trigonometric identities:

\[
\sin(A + B) = \sin(A) \cos(B) + \cos(A) \sin(B)
\]

\[
\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B)
\]

Gradient operator in cylindrical co-ordinates:

\[
\hat{T} = \hat{r} \frac{\partial T}{\partial r} + \hat{\phi} \frac{1}{r} \frac{\partial T}{\partial \phi}
\]
Problem 1 (20 Pts)
The temperature, $T(x)$, along a bar is governed by the equation,
\[
\mu \frac{d^2 T}{dx^2} = \rho(x)
\]
where $\mu$ is a constant and $\rho(x)$ is a given local applied cooling rate per unit length. The bar runs from $x = 0$ to $x = L$, with boundary conditions $T = T_0$ at $x = 0$ and $T = 2T_0$ at $x = L$. Suppose $\rho(x) = -A \sin(3\pi x/L)$, where $A$ is a constant, find the solution for $T(x)$ which satisfies the equation and boundary conditions. Leave your answer in terms of $\mu, L, T_0$, and $A$.

Problem 2 (40 Pts)
The temperature, $T(x, y; t)$, inside a rectangle is governed by the diffusion equation, \( \frac{\partial T}{\partial t} = \alpha^2 \nabla^2 T \),
where $t$ is time, $\alpha^2$ is a material constant, and $\nabla^2$ is the Laplace operator: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The rectangle has length $A$ in the x direction and $B$ in the y direction. The boundary of the rectangle is held at temperature $T_0$ at all times. Initially, the temperature profile is given by
\[
T(x, y) = T_0 + \sin(2\pi x/A) \sin(4\pi y/B).
\]
Find an expression for the temperature valid at all time. Leave your answer in terms of $\alpha^2, A, B$, and $T_0$.

Problem 3 (40 Pts)
The temperature, $T(r, \theta)$, inside a disk of radius $a$ is governed by Laplace’s equation, $\nabla^2 T = 0$. The boundary of the disk at $r = a$ is held at a temperature, $T(\theta) = T_0 \cos(2\theta) \sin(2\theta)$. Solve for $T(r, \theta)$. Leave your answer in terms of $T_0$ and $a$. 
Appendix

The Diffusion Equation

The governing equation for thermal diffusion is of the form

\[ \alpha^2 \nabla^2 T = \frac{\partial T}{\partial t}, \]  

where \( T \) is the temperature, with

\[ \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]  

\[ \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \]  

\[ \equiv \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \phi} \left( \frac{\partial^2}{\partial \phi} \right) \]

in the cartesian, cylindrical and spherical coordinates, respectively.

Some useful second order ordinary differential equations

Euler-Cauchy Equations

An Euler-Cauchy equation is of the form

\[ x^2 y'' + bxy' + cy = 0 \]  

where \( b \) and \( c \) are constant numbers. These equations can be solved by using the change of variable \( x = e^t \).

Legendre Equation

The Legendre differential equation is the second-order ordinary differential equation

\[ (1 - x^2) y'' - 2xy' + l(l+1) y = 0. \]  

The equation has two linearly independent solutions, a solution \( P_l(x) \) which is regular at finite points is called a Legendre function of the first kind, while a solution \( Q_l(x) \) which is singular at ±1 called a Legendre function of the second kind. If \( l \) is an integer, the function of the first kind reduces to a polynomial known as the Legendre polynomial. The first few Legendre polynomials
are

\[ P_0(x) = 1 \]
\[ P_1(x) = x \]
\[ P_2(x) = \frac{1}{2}(3x^2 - 1) \]
\[ P_3(x) = \frac{1}{2}(5x^3 - 3x) \]
\[ P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \]
\[ P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \]
\[ P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5). \]

If the variable \( x \) is replaced by \( \cos \theta \), then the Legendre differential equation becomes

\[
\frac{d^2y}{d\theta^2} + \cos \theta \frac{dy}{d\theta} + l(l + 1)y = 0.
\]  

(7)

Bessel Equation

The differential equation

\[
x^2 y'' + xy' + (x^2 - n^2)y = 0
\]

has two classes of solution, called the Bessel function of the first kind \( J_n(x) \) and Bessel function of the second kind \( Y_n(x) \). It follows then \( J_n(\kappa x) \) and \( Y_n(\kappa x) \) are solutions to the equation

\[
x^2 y'' + xy' + (\kappa^2 x^2 - n^2)y = 0
\]

(9)

Some useful Bessel identities are as follows:

\[
\frac{d[x^n J_n(x)]}{dx} = x^n J_{n-1}(x),
\]

\[
\int_0^c [J_n(\kappa x)]^2 x dx = \frac{c^2}{2} [J_{n+1}(\kappa c)]^2
\]

Laplace Transform

The Laplace transform of a function \( f(t) \), defined for all real numbers \( t \geq 0 \), is the function \( F(s) \), defined by

\[
F(s) = \int_0^\infty f(t)e^{-st} dt
\]
MTH PhD Qualifying Exam, Northeastern University MIE Dept. September 2019

Problem 1 (20 Pts):
A thin spherical shell of radius, $a$, has a surface mass density given by

$$\rho = \rho_0 \cos^2(\theta)$$

where $\rho_0$ is a constant with units of mass per unit area and $\theta$ is the polar angle. As usual, $\theta = 0$ and $\pi$ represent the north and south pole respectively.

a) Find the total mass of the shell in terms of $a$ and $\rho_0$.
b) Find the moment of inertia about the polar axis in terms of $a$ and $\rho_0$. Recall that the moment of inertia is the integral of the mass density times the square of the distance from the axis of rotation.

Problem 2 (40 Pts): Consider the discrete Markov process whose dynamics is given by:

$$x_{n+1} = Ax_n$$

where $A$ is the Markov matrix

$$A = \begin{bmatrix} 0.72 & 0.12 & 0.10 \\ 0.20 & 0.78 & 0.03 \\ 0.08 & 0.10 & 0.87 \end{bmatrix}$$

a) Verify explicitly that the eigenvectors and eigenvalues are $\lambda_1 = 1, e_1 = [56, 62, 82]^T; \lambda_2 = 0.77, e_2 = [9, 70, -79]^T; \lambda_3 = 0.60, e_3 = [62, -70, 7]^T$ (recall that eigenvectors are only defined up to an arbitrary overall scale factor).
b) Suppose the initial value of $x$ is $x_0 = [1, 0, 0]^T$. Find the value, $x_3$, of $x$ after three iterations of the Markov process. Hint: Use eigenvector decomposition techniques by expressing the initial state as a linear combination of the eigenvectors. **YOU MAY NOT USE YOUR SCIENTIFIC CALCULATOR TO PERFORM THE EXPLICIT MATRIX MULTIPLICATION!**

Problem 3 (40 Pts): Consider the Poisson equation with a source term of the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = xy.$$  \hspace{1cm} (1)

Solve for $u(x, y)$ over a rectangular domain $0 < x < a$, $0 < y < b$, subject to the end conditions

$$u(0, y) = u(x, b) = u(a, y) = u(x, 0) = 0.$$  \hspace{1cm} (2)

If your final solution is in the form of a series expansion, you must provide a full evaluation of the expansion coefficients.